Chapter 7

Finite Volume Method

Finite Volume Method

Conservative laws of fluid motion

Laws:

1- Conservation of mass

2- Newton's 2nd law

3- Conservation of energy

View points:

1. Lagrangian

2. Eulerian

Finite Volume Method

Conservation of mass or continuity

$$\boxed{\frac{D}{Dt}\int_{V} \rho dV = 0}$$

$$\frac{\partial}{\partial t}\int_{C.V.} \rho dV + \int_{C.S.} \rho \vec{v}.\hat{n} dA = 0$$

Using divergence theorem

$$\frac{\partial}{\partial t}\int_{C.V.} \rho dV + \int_{C.V.} \nabla.\left(\rho \vec{v}\right) dV = 0$$

$$\Rightarrow \int_{C.V.} \left[\frac{\partial \rho}{\partial t} + \nabla.\left(\rho \vec{v}\right)\right] dV = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \nabla.\left(\rho \vec{v}\right) = 0$$

Finite Volume Method

Conservation of momentum:

$$\frac{D}{Dt}\int_{V} \rho \vec{v} dV = \vec{F} = \vec{F}_c + \vec{F}_b$$

$$\boxed{\frac{D}{Dt}\int_{V} \rho \vec{v} dV = \int_{C.S.} \vec{T} dA + \int_{C.V.} \rho \vec{f} dV}$$

The stress tensor:

$$T_1 = \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3$$

$$T_2 = \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3 \longrightarrow \vec{T} = \boldsymbol{\sigma}.\hat{n} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

$$T_3 = \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3$$

Finite Volume Method

$$\frac{D}{Dt}\int_{V}\rho\vec{v}\,dV=\int_{C.S.}\vec{\sigma}.\hat{n}\,dA+\int_{C.V.}\rho\vec{f}\,dV$$

$$\frac{\partial}{\partial t}\int_{C.V.}\rho\vec{v}\,dV+\int_{C.S.}\vec{v}\left(\rho\vec{v}.\hat{n}\right)dA=\int_{C.S.}\vec{\sigma}.\hat{n}\,dA+\int_{C.V.}\rho\vec{f}\,dV$$

x- component:

$$\frac{\partial}{\partial t}\int_{C.V.}\rho u\,dV+\int_{C.S.}u\left(\rho\vec{v}.\hat{n}\right)dA=\int_{C.S.}\left(\vec{\sigma}.\hat{n}\right)_x dA+\int_{C.V.}\rho f_x\,dV$$

Finite Volume Method

Using divergence theorem

$$\frac{\partial}{\partial t}\int_{C.V.}\rho u\,dV+\int_{C.V.}\nabla.\left(\rho u\vec{v}\right)dV=\int_{C.V.}\left(\nabla.\vec{\sigma}\right)_x dV+\int_{C.V.}\rho f_x\,dV$$

$$\frac{\partial}{\partial t}\int_{C.V.}\rho u\,dV+\int_{C.V.}\nabla.\left(\rho u\vec{v}\right)dV=\int_{C.V.}\left(\nabla.\vec{\sigma}\right)_x dV+\int_{C.V.}\rho f_x\,dV$$

$$\Rightarrow\int\left[\frac{\partial\rho u}{\partial t}+\nabla.\left(\rho u\vec{v}\right)-\left(\nabla.\vec{\sigma}\right)_x-\rho f_x\right]dV=0$$

$$\Rightarrow\frac{\partial\rho u}{\partial t}+\nabla.\left(\rho u\vec{v}\right)=\left(\nabla.\vec{\sigma}\right)_x+\rho f_x$$

Finite Volume Method

Recall the material derivative:

$$\rho\frac{D\phi}{Dt}=\rho\frac{\partial\phi}{\partial t}+\rho\vec{v}.\nabla\phi$$

Adding the continuity:

$$\rho\frac{D\phi}{Dt}=\rho\frac{\partial\phi}{\partial t}+\rho\vec{v}.\nabla\phi+\left(\frac{\partial\rho}{\partial t}+\nabla.\rho\vec{v}\right)\phi$$

$$\Rightarrow\rho\frac{D\phi}{Dt}=\frac{\partial}{\partial t}(\rho\phi)+\nabla.\left(\rho\phi\vec{v}\right)$$

and

$$\left(\nabla.\vec{\sigma}\right)_x=\frac{\partial(-p+\tau_{xx})}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}$$

Finite Volume Method

therefore

$$\rho\frac{Du}{Dt}=\frac{\partial(-p+\tau_{xx})}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+S_{M_x}$$

similarly

$$\rho\frac{Dv}{Dt}=\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\left(-p+\tau_{yy}\right)}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}+S_{M_y}$$

$$\rho\frac{Dw}{Dt}=\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial(-p+\tau_{zz})}{\partial z}+S_{M_z}$$

Finite Volume Method

For Newtonian fluid

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} + \lambda\nabla.\vec{v}, \qquad \tau_{yy} = 2\mu\frac{\partial v}{\partial x} + \lambda\nabla.\vec{v}, \qquad \tau_{zz} = 2\mu\frac{\partial w}{\partial x} + \lambda\nabla.\vec{v}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \quad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \quad \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

therefore

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x} + \lambda\nabla.\vec{v}\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + S_{M_x}$$

finally

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right) + \left[\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial x}\right)\right] + \frac{\partial}{\partial x}\left(\lambda\nabla.\vec{v}\right)$$

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Finite Volume Method

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \nabla.(\mu\nabla u) + \left[\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial x}\right)\right] + \frac{\partial}{\partial x}\left(\lambda\nabla.\vec{v}\right)$$

then

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \nabla.(\mu\nabla u) + S_{M_x}$$

similarly

$$\rho\frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \nabla.(\mu\nabla v) + S_{M_y}$$

$$\rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \nabla.(\mu\nabla w) + S_{M_z}$$

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Finite Volume Method

Energy Equation

One can derive

$$\rho\frac{De}{Dt} = -p\nabla.\vec{v} + \nabla.(k\nabla T) + \Phi + S_e$$

where

$$\Phi = \mu\left\{2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2\right\} + \lambda\left(\nabla.\vec{v}\right)^2$$

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Finite Volume Method

Conservation form of the governing equation of fluid flow

Mass:

$$\frac{\partial \rho}{\partial t} + \nabla.\left(\rho\vec{v}\right) = 0$$

$$\frac{\partial(\rho u)}{\partial t} + \nabla.\left(\rho u\vec{v}\right) = -\frac{\partial p}{\partial x} + \nabla.(\mu\nabla u) + S_{M_x}$$

$$\frac{\partial(\rho v)}{\partial t} + \nabla.\left(\rho v\vec{v}\right) = -\frac{\partial p}{\partial y} + \nabla.(\mu\nabla v) + S_{M_y}$$

$$\frac{\partial(\rho w)}{\partial t} + \nabla.\left(\rho w\vec{v}\right) = -\frac{\partial p}{\partial z} + \nabla.(\mu\nabla w) + S_{M_z}$$

Energy Eq.

$$\frac{\partial(\rho e)}{\partial t} + \nabla.\left(\rho e\vec{v}\right) = -p\nabla u + \nabla.(k\nabla T) + \Phi + S_e$$

Where

$$p = p(\rho, T), \qquad e = e(\rho, T)$$

General form

$$\boxed{\frac{\partial}{\partial t}(\rho\phi) + \nabla.\left(\rho\phi\vec{v}\right) = \nabla.(\Gamma\nabla\phi) + S_\phi}$$

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Finite Volume Method

General form

$$\boxed{\frac{\partial}{\partial t}(\rho\phi) + \nabla.\left(\rho\phi\vec{v}\right) = \nabla.(\Gamma\nabla\phi) + S_\phi}$$

S_ϕ	ϕ	Γ	Equation
٠	١	٠	Continuity
$-\vec{\nabla}p + S_v$	$\vec{V}$	μ	momentum
$-\rho\vec{\nabla}\cdot\vec{V} + \Phi_D + S_i$	T	k	Energy Eq.

Finite Volume Method

The control volume integration, which forms the key step of the finite volume method that distinguishes it from all other CFD techniques, yields the following form:

$$\int_A \rho\phi v\cdot\hat{n}\, dA = \int_A \Gamma\nabla\phi\cdot\hat{n}\, dA + \int_{\forall} q_\phi d\forall$$

Integration of this equation:

$$\int_{\forall}\frac{\partial(\rho\phi)}{\partial t}d\forall + \int_{\forall}\nabla\cdot(\rho\phi u)d\forall = \int_{\forall}\nabla\cdot(\Gamma\nabla\phi)d\forall + \int_{\forall}S_\phi d\forall$$

Use Gusse Divergence

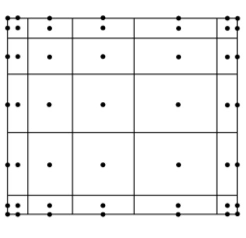
$$\frac{\partial}{\partial t}\left(\int_{\forall}\rho\phi d\forall\right) + \int_A n\cdot(\rho\phi u)dA = \int_A n\cdot(\Gamma\nabla\phi)dA + \int_{\forall}S_\phi d\forall$$

For Steady state

$$\int_A n\cdot(\rho\phi u)dA = \int_A n\cdot(\Gamma\nabla\phi)dA + \int_{\forall}S_\phi d\forall$$

Finite Volume Method

Kinds of grid generation for Finite Volume Method



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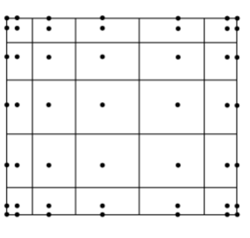
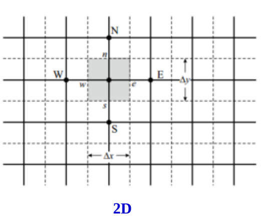


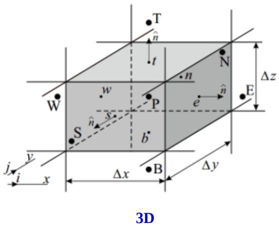
A

Approximations of Finite Volume Method

Approximation of surface integration



2D



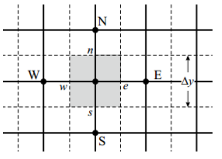
3D

$$\boxed{\int_A f dA = \sum_k \int_{A_k} f dA}$$

Approximations of Finite Volume Method

Midpoint rule

$$\boxed{F_e = \int_{A_e} f dA = \bar{f}_e A_e \approx f_e A_e}$$

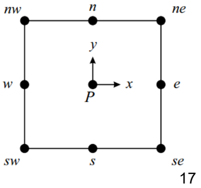


Trapezoid

$$\boxed{F_e = \int_{A_e} f dA = \frac{A_e}{2}(f_{ne} + f_{se})}$$

Simpson's rule

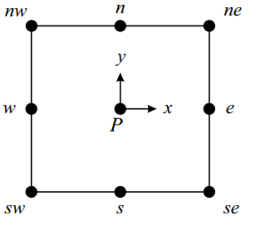
$$\boxed{F_e = \int_{A_e} f\, dA \approx \frac{A_e}{6}(f_{ne} + 4f_e + f_{se})}$$



Approximations of Finite Volume Method

approximation of volume integration

$$Q_P = \int_{\forall} q d\forall = \bar{q}\Delta\forall \approx q_P \Delta\forall$$



Approximations of Finite Volume Method

Bi- quadratic $q(x,y) = a_o + a_1 x + a_2 y + a_3 x^2 + a_4 y^2 + a_5 xy + a_6 x^2 y + a_7 x y^2 + a_8 x^2 y^2$

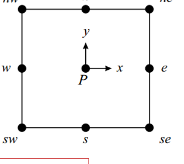
$$Q_P = \int_{\forall} q\, d\forall = \int_{-\frac{\Delta y}{2}}^{\frac{\Delta y}{2}} \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} q(x,y) dx dy$$

$$= \Delta x \Delta y \left(a_0 + \frac{a_3}{12}(\Delta x)^2 + \frac{a_4}{12}(\Delta y)^2 + \frac{a_8}{144}(\Delta x)^2(\Delta y)^2\right)$$

$$q(0,0) = q_P \qquad q\left(\frac{\Delta x}{2}, \frac{\Delta y}{2}\right) = q_{ne} \qquad q\left(\frac{\Delta x}{2}, 0\right) = q_e$$

$$q\left(\frac{\Delta x}{2}, -\frac{\Delta y}{2}\right) = q_{se} \qquad q\left(0, -\frac{\Delta y}{2}\right) = q_s \qquad q\left(-\frac{\Delta x}{2}, -\frac{\Delta y}{2}\right) = q_{sw}$$

$$q\left(-\frac{\Delta x}{2}, 0\right) = q_w \qquad q\left(-\frac{\Delta x}{2}, \frac{\Delta y}{2}\right) = q_{nw} \qquad q\left(0, \frac{\Delta y}{2}\right) = q_n$$



$$\boxed{Q_P = \frac{\Delta x^2}{36}(16q_P + 4q_s + 4q_n + 4q_w + 4q_e + q_{se} + q_{sw} + q_{ne} + q_{nw})}$$

FVM for one dimensional steady state diffusion

One dimensional steady- state diffusion problem:

$$\boxed{\operatorname{div}(\Gamma \operatorname{grad} \phi) + S_\phi = 0}$$

Using the divergence problem:

$$\boxed{\int_{CV} \operatorname{div}(\Gamma \operatorname{grad} \phi) dV + \int_{CV} S_\phi dV = \int_A \mathbf{n} \cdot (\Gamma \operatorname{grad} \phi) dA + \int_{CV} S_\phi dV = 0}$$

This is the key advantage of FVM

FVM for one dimensional steady state diffusion

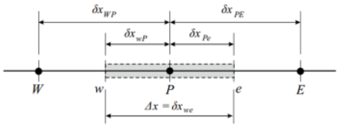
The application of the finite volume method to the solution of simple diffusion problems involving conductive heat transfer is presented:

$$\boxed{\frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right)+S=0}$$

the source term S may be a function of the dependent variable. In such cases the finite volume method approximates the source term by means of a linear form: $S = S_u + S_P T_P$

Step 1 : Grid generation

The first step in the finite volume method is to divide the domain into discrete control volumes.



FVM for one dimensional steady state diffusion

Step 2: Discretization

$$\int_{\Delta V}\frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right)dV+\int_{\Delta V}S\,dV=\left(\Gamma A\frac{d\phi}{dx}\right)_e-\left(\Gamma A\frac{d\phi}{dx}\right)_w+\bar{S}\Delta V=0$$

Linear approximations seem to be the obvious and simplest way of calculating interface values and the gradients. This practice is called central differencing. In a uniform grid linearly interpolated value for Γ is given by

$$\Gamma_w=\frac{\Gamma_W+\Gamma_P}{2}$$

$$\Gamma_e=\frac{\Gamma_P+\Gamma_E}{2}$$

and

$$\left(\Gamma A\frac{d\phi}{dx}\right)_e=\Gamma_e A_e\left(\frac{\phi_E-\phi_P}{\delta x_{PE}}\right)$$

$$\left(\Gamma A\frac{d\phi}{dx}\right)_w=\Gamma_w A_w\left(\frac{\phi_P-\phi_W}{\delta x_{WP}}\right)$$

FVM for one dimensional steady state diffusion

In practical situations, as illustrated later, the source term S may be a function of the dependent variable. In such cases the finite volume method approximates the source term by means of a linear form:

$$\bar{S}\Delta V=S_u+S_p\phi_P$$

Therefore

$$\Gamma_e A_e\left(\frac{\phi_E-\phi_P}{\delta x_{PE}}\right)-\Gamma_w A_w\left(\frac{\phi_P-\phi_W}{\delta x_{WP}}\right)+(S_u+S_p\phi_P)=0$$

This can be rearranged as

$$\boxed{\left(\frac{\Gamma_e}{\delta x_{PE}}A_e+\frac{\Gamma_w}{\delta x_{WP}}A_w-S_p\right)\phi_P=\left(\frac{\Gamma_w}{\delta x_{WP}}A_w\right)\phi_W+\left(\frac{\Gamma_e}{\delta x_{PE}}A_e\right)\phi_E+S_u}$$

FVM for one dimensional steady state diffusion

Which can be written as

$$\boxed{a_P\phi_P=a_W\phi_W+a_E\phi_E+S_u}$$

Where

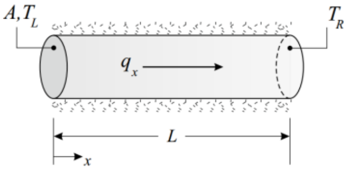
a_W	a_E	a_P
$\frac{\Gamma_w}{\delta x_{WP}}A_w$	$\frac{\Gamma_e}{\delta x_{PE}}A_e$	$a_W+a_E-S_P$

Step 3: Solution of equations

FVM for one dimensional steady state diffusion

Example

Figure shows a cylinder of thickness *L*=2 cm with constant thermal conductivity *k=0.5* W/m.K and uniform heat generation *q*=1000 kW/m3. The faces left and right are at temperatures of 100°C and 200°C respectively.



FVM for one dimensional steady state diffusion

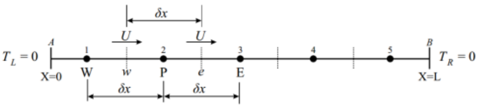
Solution

The governing equation is: $\frac{d}{dx}\left(k\frac{dT}{dx}\right)+q'''=0$

For 5 node $\quad \delta x=\frac{L}{N}=\frac{0.02}{5}=0.004\text{ m}$

Integration of the equation over a control volume

$$\int_{\forall}\frac{d}{dx}\left(k\frac{dT}{dx}\right)d\forall+\int_{\forall}q'''d\forall=0$$



FVM for one dimensional steady state diffusion

Thus

$$\int_{\forall}\frac{d}{dx}\left(k\frac{dT}{dx}\right)d\forall+\int_{\forall}q'''d\forall=0 \xrightarrow{d\forall = A dx} \left[\left(kA\frac{dT}{dx}\right)_e-\left(kA\frac{dT}{dx}\right)_w\right]+q'''\Delta v=0$$

$$\longrightarrow \left[k_eA\frac{T_E-T_P}{\delta x}-k_wA\frac{T_P-T_W}{\delta x}\right]+q'''A\delta x=0$$

The above equation can be rearranged as

$$\boxed{\left(\frac{k_eA}{\delta x}+\frac{k_wA}{\delta x}\right)T_P=\left(\frac{k_wA}{\delta x}\right)T_W+\left(\frac{k_wA}{\delta x}\right)T_E+q'''A\delta x}$$

This equation is written in the general form of

$$\boxed{a_PT_P=a_WT_W+a_ET_E+S_u}$$

FVM for one dimensional steady state diffusion

Since $(k_e=k_w=k)$ we have the following coefficients:

a_W	a_E	a_P	S_P	S_u
$\frac{kA}{\delta x}$	$\frac{kA}{\delta x}$	$a_W+a_E-S_P$	0	$qA\delta x$

Equation $a_PT_P=a_WT_W+a_ET_E+S_u$ valid for control volumes at nodal points 2, 3 and 4. To incorporate the boundary conditions at nodes 1 and 5 we apply the linear approximation for temperatures between a boundary point and the adjacent nodal point. At node 1 the temperature at the west boundary is known. Integration of equation at the control volume surrounding node 1 gives:

$$\left[\left(kA\frac{dT}{dx}\right)_e-\left(kA\frac{dT}{dx}\right)_w\right]+q'''\Delta v=0$$

$$\left[k_eA\frac{T_E-T_P}{\delta x}-k_LA\frac{T_P-T_L}{\frac{\delta x}{2}}\right]+q'''A\delta x=0$$

FVM for one dimensional steady state diffusion

The last equation can be rearranged, using $(k_e = k_w = k)$, to yield the discretized equation for boundary node 1:

$$a_P T_P = a_W T_W + a_E T_E + S_u$$

Where

a_W	a_E	a_P	S_P	S_u
$\frac{kA}{\delta x}$	0	$a_W + a_E - S_P$	$-\frac{2kA}{\delta x}$	$qA\delta x + \frac{2kA}{\delta x} T_B$

And for boundary node 5:

$$a_P T_P = a_W T_W + a_E T_E + S_u$$

Where

a_W	a_E	a_P	S_P	S_u
$\frac{kA}{\delta x}$	0	$a_W + a_E - S_P$	$-\frac{2kA}{\delta x}$	$qA\delta x + \frac{2kA}{\delta x} T_B$

FVM for one dimensional steady state diffusion

Substitution of numerical values for A=1, k=0.5, q=1000 and δx=0.004 everywhere gives the coefficients of the discretized equations summarized in Table:

a_P	S_u	S_P	a_E	a_W	گره
375	$4000 + 250T_L$	−250	125	0	1
250	4000	0	125	125	2
250	4000	0	125	125	3
250	4000	0	125	125	4
375	$4000 + 250T_B$	−250	0	125	5

FVM for One dimensional steady state diffusion

Given directly in matrix form the equations are:

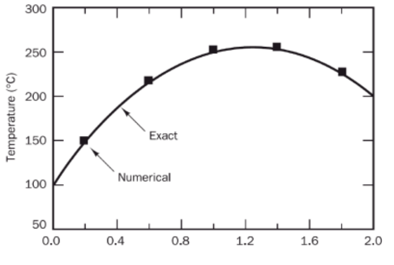
$$\begin{bmatrix} 375 & -125 & 0 & 0 & 0 \\ -125 & 250 & -125 & 0 & 0 \\ 0 & -125 & 250 & -125 & 0 \\ 0 & 0 & -125 & 250 & -125 \\ 0 & 0 & 0 & -125 & 375 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 29000 \\ 4000 \\ 4000 \\ 4000 \\ 54000 \end{bmatrix}$$

The solution to the above set of equations is:

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 150 \\ 218 \\ 254 \\ 258 \\ 230 \end{bmatrix}$$

FVM for one dimensional steady state diffusion

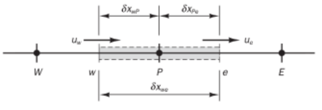
Comparison with the analytical solution



FVM for Steady one dimensional convection and diffusion

In the absence of sources, steady convection and diffusion of a property φ in a given one-dimensional flow field u is governed by

$$\boxed{\frac{d}{dx}(\rho u \phi) = \frac{d}{dx}\left(\Gamma \frac{d\phi}{dx}\right)}$$



Integration of transport equation over the control volume gives

$$(\rho u \phi)_e - (\rho u \phi)_w = \left(\Gamma \frac{d\phi}{dx}\right)_e - \left(\Gamma \frac{d\phi}{dx}\right)_w$$

FVM for Steady one dimensional convection and diffusion

$$(\rho u \phi)_e - (\rho u \phi)_w = \left(\Gamma \frac{d\phi}{dx}\right)_e - \left(\Gamma \frac{d\phi}{dx}\right)_w$$

$$\left|\begin{array}{l} \phi_e = \dfrac{\phi_E + \phi_P}{2} \\ \phi_w = \dfrac{\phi_P + \phi_W}{2} \end{array}\right. \longrightarrow (\rho u)_e \frac{(\phi_E + \phi_P)}{2} - (\rho u)_w \frac{(\phi_P + \phi_W)}{2} = \frac{\Gamma_e(\phi_E - \phi_P)}{\delta x_e} - \frac{\Gamma_w(\phi_P - \phi_W)}{\delta x_w}$$

or

$$\boxed{\left[\frac{(\rho u)_e}{2} - \frac{(\rho u)_w}{2} + \frac{\Gamma_e}{\delta x_e} + \frac{\Gamma_w}{\delta x_w}\right]\phi_P = \left[\frac{\Gamma_e}{\delta x_e} - \frac{(\rho u)_e}{2}\right]\phi_E + \left[\frac{(\rho u)_w}{2} + \frac{\Gamma_w}{\delta x_w}\right]\phi_W}$$

Note

Two variables F and D to represent the convective mass flux per unit area and diffusion conductance at cell faces

$$F = \rho u \text{ and } D = \frac{\Gamma}{\delta x}$$

FVM for Steady one dimensional convection and diffusion

Identifying the coefficients of ϕ_W and ϕ_E as a_W and a_E, the central differencing expressions for the discretized convection–diffusion equation are:

$$\boxed{a_P \phi_P = a_E \phi_E + a_W \phi_W}$$

Where

a_W	a_E	a_P
$D_w + \frac{F_w}{2}$	$D_e - \frac{F_e}{2}$	$a_W + a_E + (F_e - F_w)$

FVM for Steady one dimensional convection and diffusion

Example

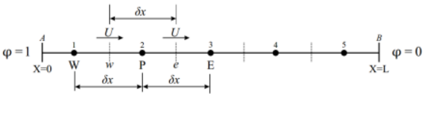
A property φ is transported by means of convection and diffusion through the one-dimensional domain sketched in Figure. the boundary conditions are

$x = 0 \rightarrow \phi_0 = 1$

$x = L \rightarrow \phi_L = 0$

Using five equally spaced cells and the central differencing scheme for convection and diffusion, calculate the distribution of φ as a function of x for (i) Case 1: $u = 0.1$ m/s, (ii) Case 2: $u = 2.5$ m/s, and compare the results with the analytical solution

$$\frac{\phi - \phi_0}{\phi_L - \phi_0} = \frac{\exp(\rho u x/\Gamma) - 1}{\exp(\rho u L/\Gamma) - 1}$$



FVM for Steady one dimensional convection and diffusion

Solution

The following data apply: $\Gamma = 0.1\,\frac{\text{kg}}{\text{m}\cdot\text{s}}$ $\quad \rho = 1\,\frac{\text{kg}}{\text{m}^3}$ $\quad L = 1\text{m}$ $\quad D = \frac{\Gamma}{\delta x}$ $\quad F = \rho u$

Integration of the equation over a control volume at nodal points 2, 3 and 4:

$$\int (\rho u \phi) dA = \int (\Gamma \frac{d\phi}{dx}) dA \longrightarrow (F\phi)_e - (F\phi)_w = (\Gamma \frac{d\phi}{dx})_e - (\Gamma \frac{d\phi}{dx})_w$$

$$\boxed{F_e \frac{\phi_E + \phi_P}{2} - F_w \frac{\phi_P + \phi_W}{2} = D_e(\phi_E - \phi_P) - D_w(\phi_P - \phi_W)}$$

$$\longrightarrow \left[\left(D_w + \frac{F_w}{2}\right) + \left(D_e - \frac{F_e}{2}\right) + (F_e - F_w)\right]\phi_P = (D_w + \frac{F_w}{2})\phi_W + (D_e - \frac{F_e}{2})\phi_E$$

Where

$$a_W = D_w + \frac{F_w}{2}, \quad a_E = D_e - \frac{F_e}{2}, \quad a_P = a_W + a_E + (F_e - F_w) - S_P$$

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FVM for Steady one dimensional convection and diffusion

We integrate governing equation and use central differencing for both the diffusion terms and the convective flux through the east face of cell 1. The value of φ is given at the west face of this cell $\phi_w = \phi_A = 1$ so we do not need to make any approximations in the convective flux term at this boundary.

This yields the following equation for node 1:

$$\boxed{\frac{F_e}{2}(\phi_P + \phi_E) - F_A\phi_A = D_e(\phi_E - \phi_P) - D_A(\phi_P - \phi_A)}$$

For control volume 5, the *φ-value* at the east face is zero. We obtain

$$\boxed{F_B\phi_B - \frac{F_w}{2}(\phi_P + \phi_W) = D_B(\phi_B - \phi_P) - D_w(\phi_P - \phi_W)}$$

Where

$$F_A = F_B = F, \qquad D_A = D_B = \frac{\Gamma}{\frac{\delta x}{2}} = 2D$$

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FVM for Steady one dimensional convection and diffusion

(i) Case 1

$u = 0.1$ m/s: $F = \rho u = 0.1$, $D = \Gamma/\delta x = 0.1/0.2 = 0.5$ gives the coefficients as summarized in Table

a_P	S_u	S_P	a_E	a_W	گره
1.55	$1.1\phi_A$	−1.1	0.45	0	1
1	0	0	0.45	0.55	2
1	0	0	0.45	0.55	3
1	0	0	0.45	0.55	4
1.55	$0.9\phi_B$	−0.9	0.0	0.55	5

The solution to the above system is

$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{bmatrix} = \begin{bmatrix} 0.9421 \\ 0.8006 \\ 0.6276 \\ 0.4163 \\ 0.1579 \end{bmatrix}$$

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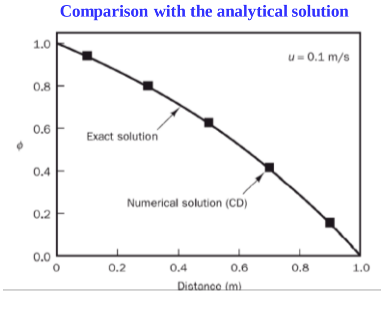
FVM for Steady one dimensional convection and diffusion

the exact solution of the problem: $\phi(x) = \frac{2.7183 - \exp(x)}{1.7183}$

Comparison with the analytical solution



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FVM for Steady one dimensional convection and diffusion

(ii) Case 2

$u = 2.5$ m/s: $F = \rho u = 2.5$, $D = \Gamma/\delta x = 0.1/0.2 = 0.5$ gives the coefficients as summarized in Table

a_P	S_u	S_P	a_E	a_W	گره
2.75	$3.5\phi_A$	−3.5	−0.75	0	1
1	0	0	−0.75	1.75	2
1	0	0	−0.75	1.75	3
1	0	0	−0.75	1.75	4
0.25	$-1.5\phi_B$	1.5	0	1.75	5

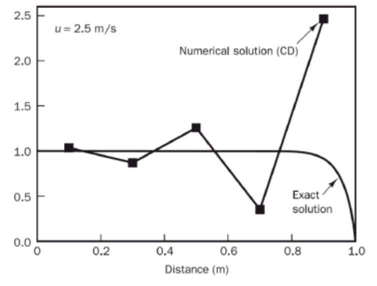
The solution to the above system is

$$\begin{bmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\\ \phi_5\end{bmatrix} = \begin{bmatrix}1.0356\\ 0.8694\\ 1.2573\\ 0.3521\\ 2.4644\end{bmatrix}$$

FVM for Steady one dimensional convection and diffusion

the exact solution of the problem: $\phi(x) = 1 + \dfrac{1 - \exp(25x)}{7.2 \times 10^{10}}$

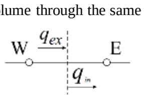
Comparison with the analytical solution



Four Properties of FVM

1. Conservativeness

To ensure conservation of φ for the whole solution domain the flux of φ leaving a control volume across a certain face must be equal to the flux of φ entering the adjacent control volume through the same face.



$$\boxed{q_{ex} = q_{in}}$$

2.Positive Coefficient

$$\boxed{a_P\phi_P = a_E\phi_E + a_W\phi_W}$$

Four Properties of FVM

3. Boundedness

The discretized equations at each nodal point represent a set of algebraic equations that needs to be solved. Normally iterative numerical techniques are used to solve large equation sets. These methods start the solution process from a guessed distribution of the variable φ and perform successive updates until a converged solution is obtained. Scarborough (1958) has shown that a sufficient condition for a convergent iterative method can be expressed in terms of the values of the coefficients of the discretized equations:

$$\boxed{\frac{\sum|a_{nb}|}{|a'_P|} \begin{cases} \le 1 \text{ at all nodes} \\ < 1 \text{ at one node at least} \end{cases}}$$

Here a′P is the net coefficient of the central node P , and the summation in the numerator is taken over all the neighboring nodes. If the differencing scheme produces coefficients that satisfy the above criterion the resulting matrix of coefficients is diagonally dominant

Four Properties of FVM

4. Summation of neighbors' coefficients

$$a_P\phi_P = a_W\phi_W + a_E\phi_E + a_S\phi_S + a_N\phi_N + S_u$$

$$\longrightarrow \quad a_P = \sum a_{nb} + \underbrace{(F_e - F_w + F_n - F_s)}_{0} - S_P$$

If established continuity equation

If $S_P = 0 \longrightarrow a_P = \sum a_{nb}$

Analytical Solution of Convection and Diffusion

Steady one dimension convection and diffusion equation:

$$\frac{d}{dx}(\rho u\phi) = \frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right) \qquad \text{B.C} \quad \begin{cases} x = 0 & \phi = \phi_P \\ x = \delta x_e & \phi = \phi_E \end{cases}$$

Analytical solution:

$$\frac{d}{dx}\left(\rho u\phi - \Gamma\frac{d\phi}{dx}\right) = 0 \longrightarrow \rho u\phi - \Gamma\frac{d\phi}{dx} = C_1 \longrightarrow \frac{d\phi}{\rho u\phi - C_1} = \frac{dx}{\Gamma}$$

thus $$\frac{1}{\rho u}\ln(\rho u\phi - C_1) = \frac{x}{\Gamma} + C_2$$

finally $$\boxed{\phi = \frac{1}{\rho u}\left[e^{\rho u(x/\Gamma + C_2)} + C_1\right]}$$

Analytical Solution of Convection and Diffusion

from B.C

$x = 0 \qquad \phi = \phi_P \Rightarrow \phi_P = \frac{1}{\rho u}(e^{\rho u C_2} + C_1)$

$x = \delta x_e \qquad \phi = \phi_E \Rightarrow \phi_E = \frac{1}{\rho u}(e^{\rho u(\delta x_e/\Gamma + C_2)} + C_1)$

$$\begin{cases} C_1 = \dfrac{\rho u}{e^{(\rho u\delta x_e)/\Gamma} - 1}\left[e^{(\rho u\delta x_e)/\Gamma}\phi_P - \phi_E\right] \\ C_2 = \dfrac{1}{\rho u}\ln\left[\dfrac{\rho u(\phi_E - \phi_P)}{e^{(\rho u\delta x_e)/\Gamma} - 1}\right] \end{cases}$$

$$\longrightarrow \begin{cases} C_1 = \dfrac{\rho u}{e^{(\rho u\delta x_e)/\Gamma} - 1}\left[e^{(\rho u\delta x_e)/\Gamma}\phi_P - \phi_E\right] \\ C_2 = \dfrac{1}{\rho u}\ln\left[\dfrac{\rho u(\phi_E - \phi_P)}{e^{(\rho u\delta x_e)/\Gamma} - 1}\right] \end{cases}$$

$$\boxed{\phi = \frac{1}{\rho u}\left[\frac{\rho u(\phi_E - \phi_P)}{(-1 + e^{(\rho u\delta x_e)/\Gamma})}e^{(\rho u x)/\Gamma} + \frac{\rho u(e^{(\rho u\delta x_e)/\Gamma}\phi_P - \phi_E)}{e^{(\rho u\delta x_e)/\Gamma} - 1}\right]}$$

Analytical Solution of Convection and Diffusion

Peclet Number: $\text{Pe} = \frac{F}{D} = \frac{\rho u\delta x}{\Gamma}$ if $\Gamma = \mu \longrightarrow$ Pe= cell Re number

$$\phi = \frac{(\phi_E - \phi_P)e^{(\rho u x)/\Gamma} + \phi_P e^{(\rho u\delta x_e)/\Gamma} - \phi_P + \phi_P - \phi_E}{e^{(\rho u\delta x_e)/\Gamma} - 1}$$

$$\phi = \phi_P + \frac{(\phi_E - \phi_P)(e^{(\rho u x)/\Gamma} - 1)}{e^{(\rho u\delta x_e)/\Gamma} - 1}$$

finally $$\longrightarrow \boxed{\frac{\phi - \phi_P}{\phi_E - \phi_P} = \frac{e^{\text{Pe}_e x/\delta x_e} - 1}{e^{\text{Pe}_e} - 1}}$$

For P and W:

$$\frac{\phi - \phi_W}{\phi_P - \phi_W} = \frac{e^{\frac{\text{Pe}_w x}{\delta x_w}} - 1}{e^{\text{Pe}_w} - 1}$$

Analytical Solution of Convection and Diffusion

Note

For every Pe

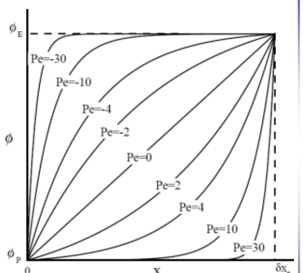
$$\boxed{\phi_E < \phi < \phi_P}$$

2. If $Pe = 0$

Equation form is heat conduction

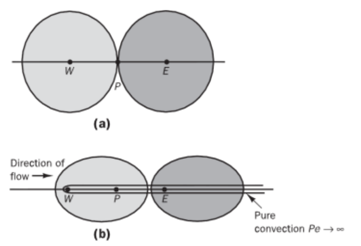
3. In mid points $\phi\left(\frac{\delta x}{2}\right) = \frac{\exp(0.5Pe - 1)}{\exp(Pe - 1)}$

$$\boxed{\begin{array}{l} Pe \to +\infty \quad \Rightarrow \quad \phi(0.5) \to 0 \\ Pe \to -\infty \quad \Rightarrow \quad \phi(0.5) \to 1 \end{array}}$$



Exact solution

Analytical Solution of Convection and Diffusion



Distribution of φ in the vicinity of two sources at different Peclet numbers:
(a) pure convection, $Pe \to 0$;
(b) diffusion and convection